Theory of a Thermal Gradient Gas Lens

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Abstract—The analysis of the gas lens presented in an earlier paper [1] is extended to a more complete ray optics description. The focal length and principal surface of the gas lens have been computed. It is found that the focal length reaches a minimum as a function of flow velocity and that the two principal surfaces coincide very nearly, making the device approximately a thin lens.

The gas lens is not free of distortions. The principal surface is not a plane, and the focal length measured from the principal surface depends on the distance of the ray from the lens axis. This dependence is rather weak for gas flow rates minimizing the focal length.

Introduction

RECENT PUBLICATIONS and experiments [1]–[3] have shown that gases can be used effectively to focus light beams.

The present paper extends the calculations of [1] to a more complete ray optics of the tubular gas lens.

This lens consists of a tube that is kept at a constant temperature higher than the temperature of the gas flowing through it. The gas heats up close to the walls. The heat penetrates radially into the gas establishing a temperature gradient that causes a corresponding density gradient. The gas density and consequently its refractive index are higher at the center of the tube and decrease toward the wall. The gas flowing through the tube acts as a lens and focuses light beams traveling through the tube.

We calculate the focal length and principal surface of the gas lens using the paraxial ray equation. We are thus able to characterize the extended focusing structure of the gas lens by an equivalent lens making it possible to predict the properties of lens combinations such as a beam waveguide composed of gas lenses by conventional optical methods without having to trace rays through the actual gas lens combinations under consideration.

Our results will show that an optimum gas velocity exists which yields not only the minimum focal length but at the same time minimizes lens distortions.

A lens distorts if its focal length depends on the ray position and if its principal surface deviates from a plane. The principal surface is defined by the points at which every straight light ray incident parallel to the optical axis has to be broken to coincide with the actual ray outside of the lens. There are two principal surfaces; one belonging to the ray incident from the left, the other to that incident from the right of the lens (Fig. 1). The lens is called thin if both principal surfaces coincide. The gas lens is nearly a thin lens.

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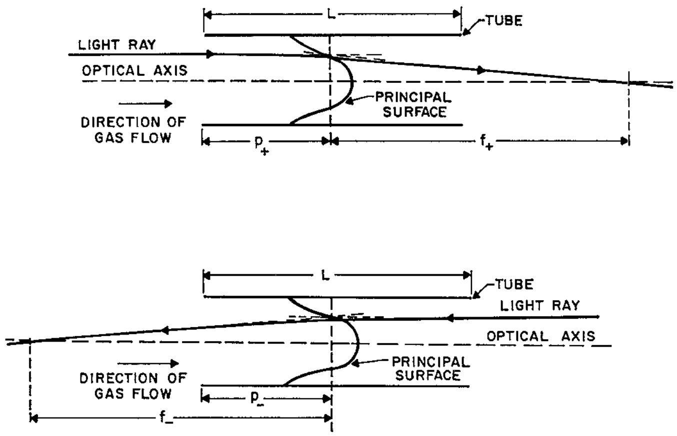

Fig. 1 (a) Definition of focal length f_+ and principle surface p_+ for rays traveling with the gas flow. (b) Definition of focal length f_- and principal surface p_- for rays traveling against the gas flow.

The focal length is the distance from the intersection between the principal surface and a straight light ray entering the lens parallel to its optical axis to the point at which this ray crosses the axis (Fig. 1).

Our analysis assumes that the light ray leaves the end of the lens without any further deflection. This is an idealized assumption. The gas temperature at the end of the lens is higher than that of its surroundings, and some transition region is needed to pass the light ray on to the surrounding medium. For measurement purposes the gas lens may be terminated by a glass plate. The light ray will then suffer some refraction in going from the warm gas in the tube, through the glass plate, into the cooler air. However, using air and temperature difference of $\Delta T = 50°C$ above room temperature, the change of the angle α between the ray and the tube axis amounts only to (provided $\alpha \ll 1$)

$$\Delta\alpha = \alpha(n-1)\frac{\Delta T}{T} = 4.5\ 10^{-5}\alpha$$

and can, therefore, be neglected.

The gas velocity of the lens will be given by a dimensionless quantity $[v_0/V(L)]$, with v_0 being the gas velocity on the axis of the tube and $V(L)$ given by

$$V(L) = \frac{kL}{a^2\rho c_p} \tag{1}$$

where

k = heat conductivity of the gas
ρ = gas density
c_p = specific heat at constant pressure
a = tube radius
L = tube length

The velocity distribution v in the tube has the parabolic profile of a viscous fluid in laminar flow

$$v = v_0(1 - x^2) \tag{2}$$

where $x = r/a$ and r = radial distance measured from tube axis. The on-axis gas velocity is related to the flow rate F (volume/time) by

$$v_0 = \frac{2}{\pi} \frac{F}{a^2}.$$

The Ray Equation

The equation of a light ray described by a position vector $\boldsymbol{r}$ is [4]

$$\frac{d}{ds}\left(n \frac{d\boldsymbol{r}}{ds}\right) = \nabla n. \tag{3}$$

s is a length coordinate measured along the light ray and n is the index of refraction of the medium. In case of our gas lens, n is very close to 1, so that we replace $n = 1$ on the left-hand side of the equation. Next we replace $ds = dz$ because the rays of interest run very closely parallel to the axis of the lens which has been chosen as the z-axis.[1] We use cylindrical polar coordinates r, ϕ, z and assume rotational symmetry around the z-axis, $\partial/\partial\phi = 0$. With the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ in the direction of the r and z coordinates, respectively, we can write

$$\boldsymbol{r} = \boldsymbol{e}_1 r + \boldsymbol{e}_3 z. \tag{4}$$

Substituting (4) into (3) yields, approximately,

$$\frac{d^2 r}{dz^2} = \frac{\partial n}{\partial r}. \tag{5}$$

Equation (5) will be the starting point for the ray optics of the gas lens.

The temperature dependence of the index of refraction is given by

$$n = (n_a - 1)\frac{T_a}{T}; \tag{6}$$

n_a is the index of refraction measured at a temperature T_a which should be chosen as an average temperature of the gas. We obtain from (6)

$$\frac{\partial n}{\partial r} = -(n_a - 1)\frac{T_a}{T^2}\frac{\partial T}{\partial r}. \tag{7}$$

Since the absolute temperature in the tube changes only slightly throughout the gas, we write

$$\frac{\partial n}{\partial r} = -(n_a - 1)\frac{1}{T_a}\frac{\partial T}{\partial r}. \tag{8}$$

[1] The error resulting from this approximation is discussed in Appendix II.

Introducing

$$x = \frac{r}{a} \tag{9a}$$

and

$$\zeta = \sigma \frac{z}{a} \tag{9b}$$

with

$$\sigma = \frac{k}{a v_0 \rho c_p} \tag{9c}$$

we obtain from (5) and (8)

$$\frac{d^2 x}{d\zeta^2} = -\frac{(n_a - 1)}{\sigma^2 T_a}\frac{\partial T(x, z)}{\partial x}. \tag{10}$$

The temperature distribution $T(x, z)$ in the gas is given by [5]

$$T(x, z) = T_w + 2\Delta T \sum_{m=0}^{\infty} \frac{R_m(x)}{\beta_m \left(\frac{\partial R_m}{\partial \beta}\right)_{x=1}} e^{-\beta_m^2 \zeta} \tag{11}$$

$$\Delta T = T_w - T_0 \tag{12}$$

where T_w = wall temperature and T_0 = (constant) temperature of gas at tube input. Curves of the temperature distribution have been shown in [1]. A discussion of the R-functions and their eigenvalues β_m is given in Appendix I.

Focal Length and Principal Surface

The position of the ray $x(u)$ and its slope $x'(u)$ were computed from (10) by numerical machine calculations using the initial condition $x'(u) = 0$ [$u = \sigma(L/a)$ and $dx/d\zeta = x'$]. Once these quantities are known, the focal length f_+ and principal surface p_+ of the lens can be computed (Fig. 1)

$$\frac{\sigma}{a} f_+ = -\frac{x(o)}{x'(u)} \tag{13}$$

$$\frac{\sigma}{a} p_+ = u + \frac{x(o) - x(u)}{x'(u)}. \tag{14}$$

The index + was added to indicate that the quantities belong to rays traveling in the positive direction or with the gas stream. Similarly, one can compute $x'(o)$ with the initial condition $x'(u) = 0$ and obtain focal length f_- and principal plane p_- for rays traveling in the opposite direction as the gas stream.

$$\frac{\sigma}{a} f_- = \frac{x(u)}{x'(o)} \tag{15}$$

$$\frac{\sigma}{a} p_- = \frac{x(u) - x(o)}{x'(o)}. \tag{16}$$

These equations allow us to present f and p as functions of the normalized tube length u. However, it is more instructive to plot f and p for a fixed tube length and variable gas velocity v_0. We normalize the gas velocity with respect to $V(L)$ of (1) to obtain the dimensionless quantity.

$$\frac{v_0}{V(L)} = \frac{a}{\sigma L} = \frac{1}{u}. \tag{17}$$

We can now write

$$\frac{f}{L} = \frac{v_0}{V(L)} \frac{\sigma}{a} f \tag{18}$$

and a similar expression for p/L.

Equations (10) and (11) show that $dx/d\zeta$ depends on a quantity

$$D = \frac{(n_a - 1)\Delta T}{\sigma^2 T_a}.$$

In a presentation which shows f and p as functions of $v_0/V(L)$ for constant tube length, D is not a constant because σ depends on v_0,

$$D = \left(\frac{v_0}{V(L)}\right)^2 \frac{(n_2 - 1)\Delta T}{T_a} \left(\frac{L}{a}\right)^2.$$

f/L and p/L as functions of $[v_0/V(L)]$ contain the parameter

$$C\left(\frac{L}{a}\right) = (n_a - 1) \frac{\Delta T}{T_a} \left(\frac{L}{a}\right)^2. \tag{19}$$

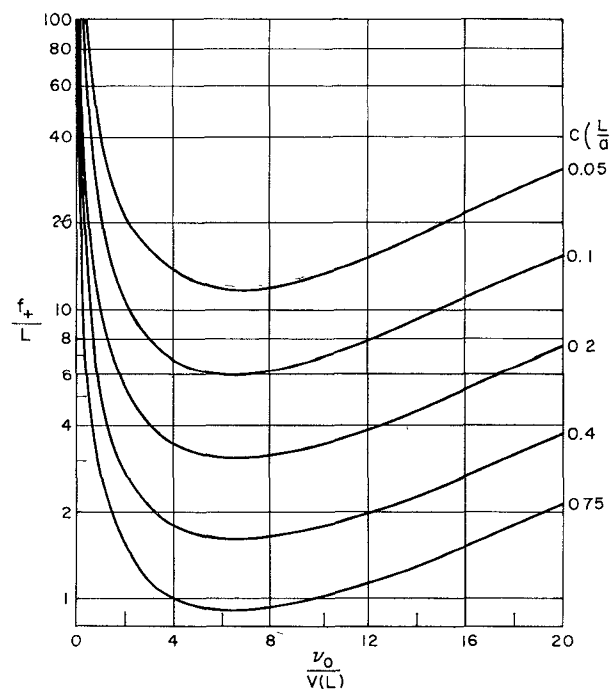

Fig. 2. Normalized focal length f/L as a function of normalized gas velocity $v_0 V(L)$ for various values of $C(L/a)$ at $x(o) = 0.1$.

Discussion of Numerical Results

Figure 2 shows the normalized focal length f_+/L as a function of $v_0/V(L)$ for rays entering the tube at $x(o) = 0.1$. The focal length reaches a minimum at approximately $[v_0/V(L)] = 6.5$. This minimum can be easily explained. For zero gas velocity there is no lens action at all since the gas heats up instantaneously at $z = 0$ so that $f = \infty$. With increasing gas velocity the lens begins to function. At very high gas velocity, on the other hand, the gas passes the tube too quickly to heat up at all and again lens action becomes impossible. This consideration shows that there must be an optimum flow rate. The focal length is very nearly proportional to C^{-1} for lower values of C. This fact is useful for interpolating f/L for other values of C. A gas lens using air, a temperature difference $\Delta T = 50°C$, $L = 20$ cm, and $a = 0.3$ cm has $C(L/a) = 0.2$. That means it has an optimum focal lens of $f/L = 3$ or $f = 60$ cm.

Figure 3 presents the points $x(o) = 0.1$ of the principal surface as a function of $v_0/V(L)$. The principal surface is remarkably independent of C. The form of this curve can again easily be explained. For zero gas velocity the lens action moves all the way to the input end of the tube. With increasing gas velocity, the active portion of the lens moves further and further down into the tube as it takes longer for the gas close to the tube axis to heat up.

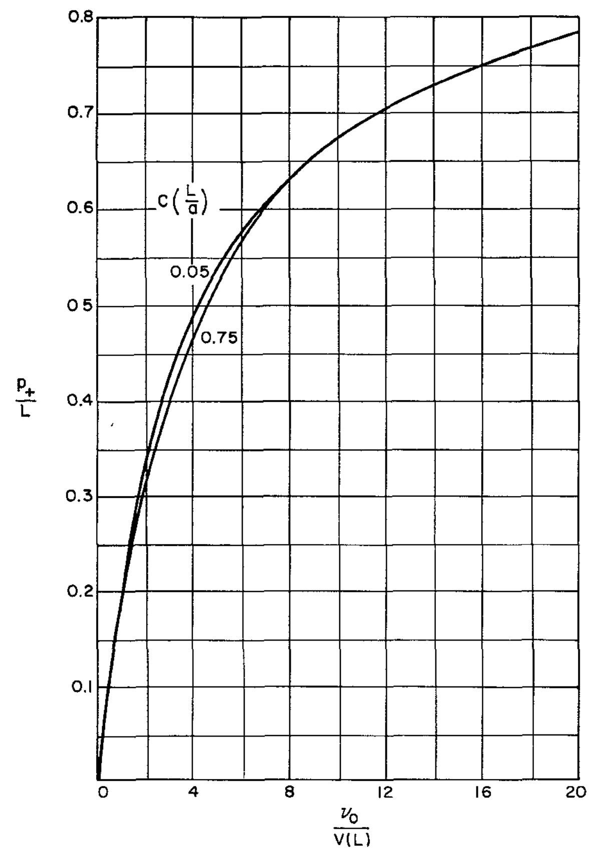

Fig. 3. Normalized position of principal surface p/L at $x(o) = 0.1$ as a function of normalized gas velocity $v_0/V(L)$ for various values of $C(L/a)$.

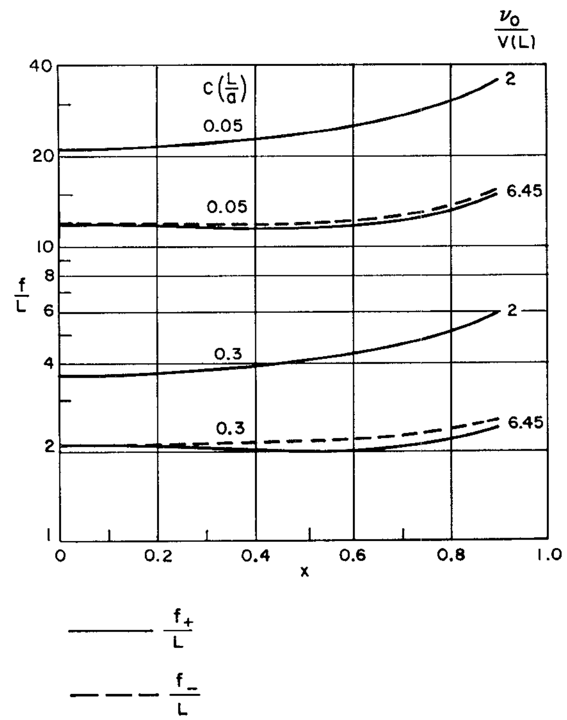

Fig. 4. Focal length as a function of the entrance position x of the ray for $v_0/V(L)=2$ and $v_0/V(L)=6.45$.

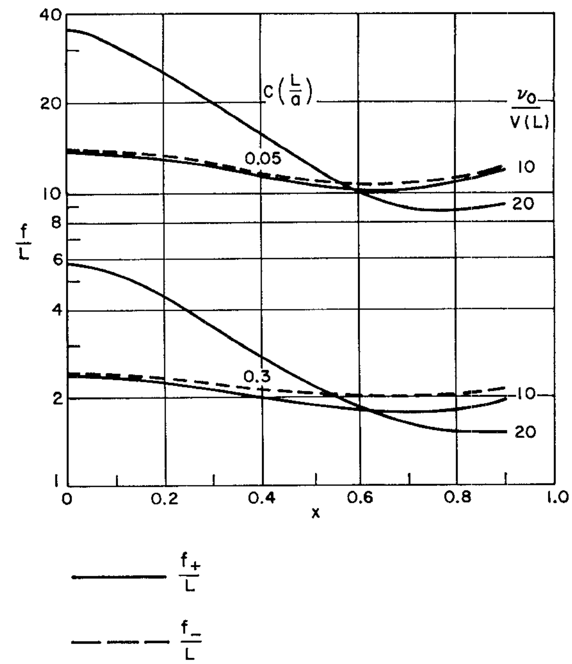

Fig. 5. Focal length as a function of the entrance position x of the ray for $v_0/V(L)=10$ and $v_0/V(L)=20$.

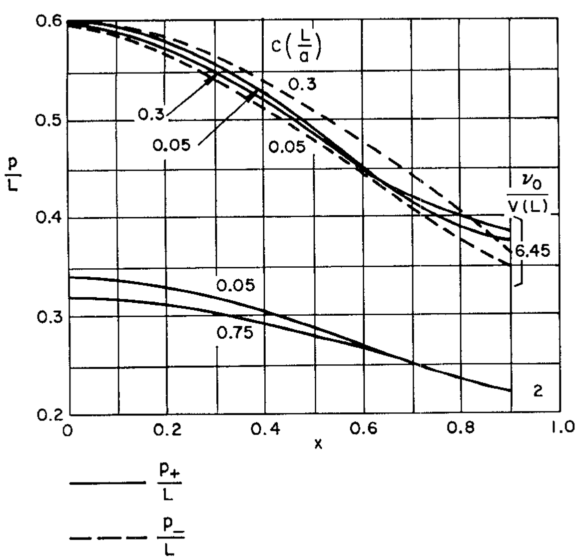

Fig. 6. Shape of the principal surface for $v_0/V(L)=2$ and $v_0/V(L)=6.45$.

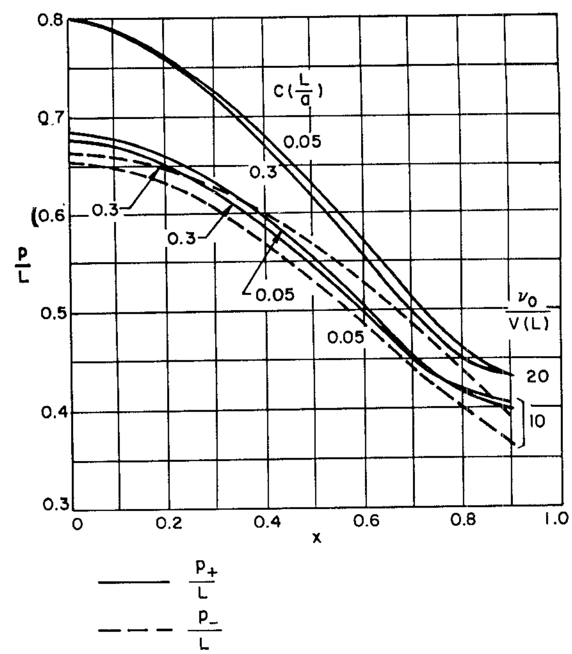

Fig. 7. Shape of the principal surface for $v_0/V(L)=10$ and $v_0/V(L)=20$.

Figures 4 and 5 show f/L as a function of the entrance position x of the ray for various values of $v_0/V(L)$. The dotted lines are the focal length f_-/L of the rays traveling in the opposite direction as the gas flow.

Of all the curves shown, the one with $v_0/V(L)=6.45$ (Fig. 4) shows the least dependence of f/L on x. It is nice that this minimum of focal length distortion occurs at the same flow velocity at which f/L has its minimum.

Finally, Figs. 6 and 7 show the shape of the principal surface for different values of $v_0/V(L)$. The dotted curves refer again to the ray traveling in the opposite direction. It can be seen that the coincidence of the two principal surfaces is not perfect. However, they are close enough to make the gas lens appear as practically a thin lens. At its optimum flow rate it is a thin lens with little distortion which is warped to fit the shape of the principal surface.

Conclusion

The discussion of the ray optics of the gas lens shows that gases can be used rather effectively to focus light and act as lenses of surprisingly short focal length and a moderate amount of lens distortion. The fact that they are optically thin lenses allows the application of the theory of thin lenses to describe their performance in an optical or light transmission system.

The present discussion does not mention convection distortions which occur under the influence of the gravitational field if the lens is operated with its axis in horizontal position.

W. H. Steiers paper [3] discusses the experimental evidence for gravitational distortions. However, his paper also shows that the theory presented here is in good agreement with experiments.

Appendix I

The R-Functions and Their Eigenvalues β_n

The R-functions are solutions of the differential equation [5], [6]

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \beta^2(1 - x^2)R = 0. \tag{20}$$

The solutions of (20) are related to Whittaker's functions $W_{\kappa,\mu}$ by

$$R = \frac{1}{x} W_{\beta/4,0}(\beta x^2). \tag{21}$$

The eigenvalue β is determined from the boundary condition

$$R(1) = 0. \tag{22}$$

To compute R we use the series expansion

$$R(x) = \sum_{v=0}^{\infty} C_{2v} x^{2v}. \tag{23}$$

For the problem of interest to us, $R(x)$ has to be an even function of x, and for that reason only even powers of x appear in (23). The normalization

$$R(0) = 1$$

requires that

$$C_0 = 1. \tag{24}$$

Substitution of (23) into (20) using (24) leads to

$$C_2 = -\frac{1}{4}\beta^2$$

and

$$C_{2v} = \frac{\beta^2}{(2v)^2}\left\{C_{2v-4} - C_{2v-2}\right\} \qquad \text{for } v \geq 2. \tag{25}$$

The fact that β^2 enters in all coefficients C_{2v} makes the determination of β_n very tedious.

A further difficulty results from the fact that the coefficients C_{2v} grow to very large values, particularly for the larger values of β_n before they decrease again. The series (23) does not converge readily for values of x close to 1. In fact, it proved impossible to compute more than the first eight R-functions from (23) on the IBM 7094 computer, even using double precision, since the absolute value of R remains between zero and one but the coefficients C_{2v} grow to values above 10^{20}. The series (23) can be used to compute R_n for x in the range $0 \leq x \leq 0.5$, since the powers of x decrease rapidly enough to keep the value of the product $C_{2v}x^{2v}$ within manageable proportions.

However, in order to cover the whole range $0 \leq x \leq 1$, it proved necessary to use another series expansion. We used

$$R(x) = \sum_{v=0}^{\infty} D_v y^v \qquad \text{with } y = 1 - x \tag{26}$$

to calculate R in the range $0.5 \leq x \leq 1$.

Equation (20) expressed in terms of y reads

$$(1 - y)\frac{d^2R}{dy^2} - \frac{dR}{dy} + \beta^2(2y - 3y^2 + y^3)R = 0. \tag{27}$$

$D_0 = 0$ has to be chosen to satisfy the boundary conditions $R(1) = 0$ at $x = 1$.

Substitution of (26) into (27) yields

$$D_2 = \frac{1}{2}D_1, \quad D_3 = \frac{1}{3}D_1, \quad D_4 = \left(\frac{1}{4} - \frac{1}{6}\beta^2\right)D_1$$

$$D_v = \frac{1}{v(v-1)}\left\{(v-1)^2 D_{v-1} - \beta^2(2D_{v-3} - 3D_{v-4} + D_{v-5})\right\}. \tag{28}$$

The eigenvalue β and the coefficient D_1 have to be chosen so that R as well as R' are continuous at $x = 0.5$, where both series expansions should coincide.

By breaking the range of x into two parts and using different series expansions to cover both parts of the range it was possible to compute the R-functions and their eigenvalues. Table I shows the first fifteen eigenvalues β_n as well as $\partial R_n/\partial\beta$ and F_n' taken at $x = 1$.

TABLE I

n	β_n	$R_n^1(1)$	$\partial R/\partial\beta x=1, \beta=\beta^n$
0	2.70436	−1.01430	−0.50090
1	6.67903	1.34924	0.37146
2	10.67338	−1.57232	−0.31826
3	14.6711	1.74600	0.28648
4	18.6699	−1.89090	−0.26449
5	22.6691	2.01647	0.24799
6	26.6686	−2.12814	−0.23491
7	30.6682	2.22038	0.22485
8	34.6679	−2.32214	−0.21548
9	38.6676	2.40274	0.20779
10	42.6667	−2.48992	−0.20108
11	46.6667	2.56223	0.19516
12	50.6667	−2.64962	−0.18988
13	54.6667	2.70216	0.18513
14	58.6667	−2.76421	−0.18083

The values of $\partial R_n/\partial\beta$ were obtained from differentiation of the series (23) and evaluating it at $x=1$. The terms of the differentiated series grow very large so that only the first eight values of $\partial R_n/\partial\beta$ could be obtained. The remaining values were calculated from the approximation [6]

$$\left(\frac{\partial R}{\partial\beta}\right)_{\substack{x=1\\ \beta=\beta_n}} = -(-1)^n \frac{\pi}{6^{2/3}\Gamma\left(\frac{2}{3}\right)\beta_n^{1/3}} \tag{29}$$

which is in good agreement with the values obtained by machine calculation for larger values of n.

Appendix II
Estimate of the Error Introduced by Setting $ds=dz$

It is

$$\frac{d^2x}{ds^2} = \frac{d^2x}{dz^2}\left[1+\left(\frac{dx}{dz}\right)^2\right]^{-2}$$

$$\approx \frac{d^2x}{dz^2}\left[1-2\left(\frac{dx}{dz}\right)^2\right]. \tag{30}$$

The slope of the ray $\dot{x}=dx/dz$ is given by

$$\dot{x} = \frac{dx}{ds}(1+\dot{x}^2)^{1/2} \approx \frac{dx}{ds}\left(1+\frac{1}{2}\dot{x}^2\right). \tag{31}$$

The derivative dx/ds is obtained from

$$\frac{dx}{ds} = \int_0^s \frac{d^2x}{ds^2}\,ds = \int_0^L \frac{d^2x}{ds^2}(1+\dot{x}^2)^{1/2}dz$$

which, with the help of (30) becomes

$$\frac{dx}{ds} = \int_0^L \left(1-\frac{3}{2}\dot{x}^2\right)\frac{d^2x}{dz^2}\,dz. \tag{32}$$

Combining (31) and (32) and replacing $\dot{x}$ by its maximum value $\dot{x}_{\max}$, we see that the relative error is

$$\frac{\Delta\dot{x}}{\dot{x}} \le 2\dot{x}_{\max}^2. \tag{33}$$

It follows from (13) that the relative error of the focal length is

$$\frac{\Delta f}{f} \le 2\dot{x}_{\max}^2. \tag{34}$$

Since the focal length is given by

$$f = -\frac{x(0)}{\dot{x}(L)}, \tag{35}$$

we have

$$\dot{x}^2 = \left(\frac{x(0)}{f}\right)^2. \tag{36}$$

The ray trajectory is a monotonic function so that $|\dot{x}(L)| = |\dot{x}_{\max}|$.

A typical gas lens has a radius $a=0.3$ cm and length $L=20$ cm. Assuming $f/L=2$ and taking $x(0)=a$, we get $\dot{x}^2_{\max}=5.62\,10^{-5}$ or

$$\frac{\Delta f}{f} \le 1.12\,10^{-4}.$$

To estimate the error introduced in the computation of the principal surface we have to obtain the relative error of the ray position $x(L)$.

$$x(L) = x(0) + \int_0^s \frac{dx}{ds}\,ds = x(0) + \int_0^L \dot{x}(z)dz$$

or using (31) and (32)

$$x(L) = x(0) + \int_0^L \left(1+\frac{1}{2}\dot{x}^2\right) \cdot\left[\int_0^z \left(1-\frac{3}{2}\dot{x}^2\right)\frac{d^2x}{dz^2}\,d\zeta\right]dz$$

$$\le x(0) + (1+2\dot{x}_{\max}^2)\int_0^L dz\int_0^z \frac{d^2x}{dz^2}\,d\zeta$$

or

$$\frac{\Delta[x(L)-x(0)]}{x(L)-x(0)} \le 2\dot{x}_{\max}^2. \tag{37}$$

From (14) we obtain

$$\Delta(L-p) \le \left|\frac{1}{\dot{x}}\Delta[x(L)-x(0)]\right| + \frac{1}{2}\left|\frac{x(L)-x(0)}{\dot{x}}\,\frac{\Delta\dot{x}}{\dot{x}}\right|$$

or with the help of (33), (37), and (14) and considering that $|\dot{x}_{\max}| = |\dot{x}(L)|$,

$$\frac{\Delta(L-p)}{L-p} \le 3\dot{x}_{\max}^2. \tag{38}$$

With the numbers of the example used in the foregoing, we get

$$\frac{\Delta(L-p)}{L-p} \le 1.7\ 10^{-4}.$$

This shows that the error caused by replacing $ds=dz$ is indeed negligible.

Acknowledgment

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